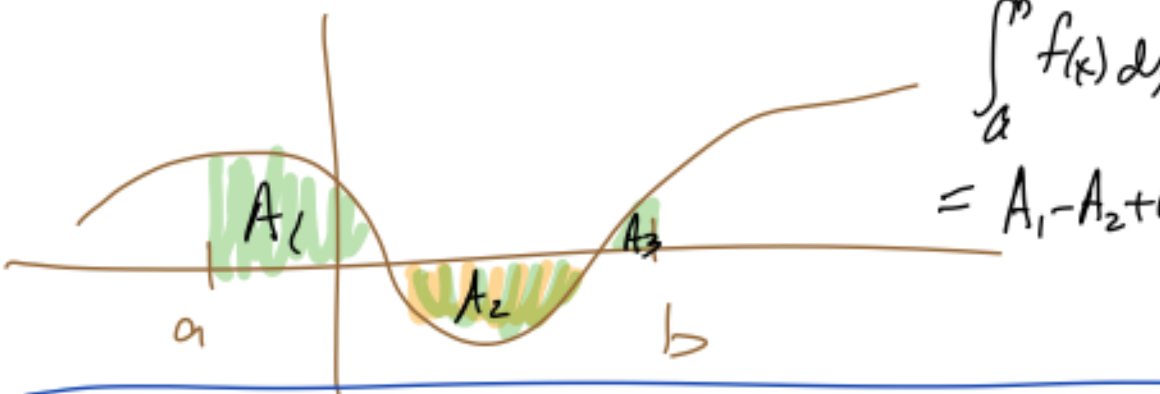
Numerical Integration.

Idea: Given a fcn $f(x)$ on an interval $[a,b]$, what is $\int_a^b f(x)\,dx$, and how can we compute it?



$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3.$$

In Calculus, we did this by finding antiderivatives, eg

$$\int_0^\pi \sin(x)\,dx = -\cos(x)\Big|_0^\pi = -\cos(\pi) - (-\cos(0))$$
$$= 2$$

$$\int_a^b \underbrace{F'(x)}_{f(x)}\,dx = F(b) - F(a).$$

But... this doesn't always work. eg $\int_0^1 e^{-x^2}\,dx = ??$

no simple antiderivative

In practice, you might need to calculate

(a) distance traveled $= \int_{t_0}^{t_f} v(t)\,dt$.

velocity.

(b) income from stocks

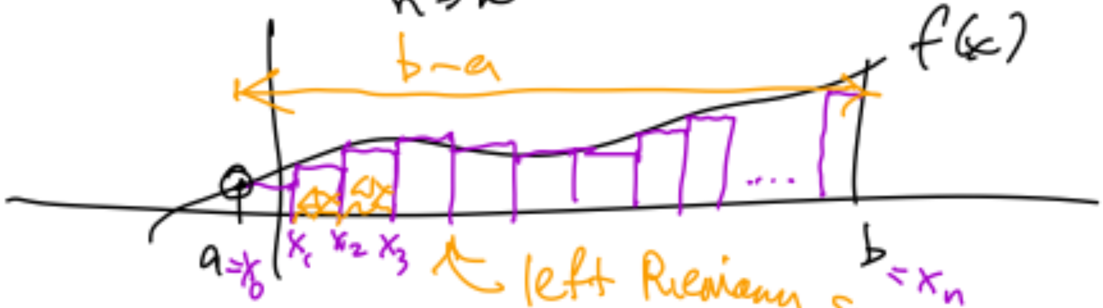
$$= \int_{t_0}^{t_f} \left(\frac{\text{daily income}}{\text{minute}}\right) dt$$

We have a fcn $f(x)$ or data

$\longrightarrow \int_a^b f(x)\,dx$.

Definition of Integral $\int_a^b f(x)\,dx$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}(\text{sum of } n \text{ rectangles})$$



$$\text{Left}(n) = \sum_{k=0}^{n-1} f(x_k)\,\Delta x \text{ , where}$$

height.

$\Delta x = \frac{b-a}{n}$, ~~$x_j = a + j\Delta x$.~~

$x_k = a + k\Delta x$
$\forall k$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \text{Left}(n).$$

(As long as f is piecewise continuous).

Right Riemann Approximation

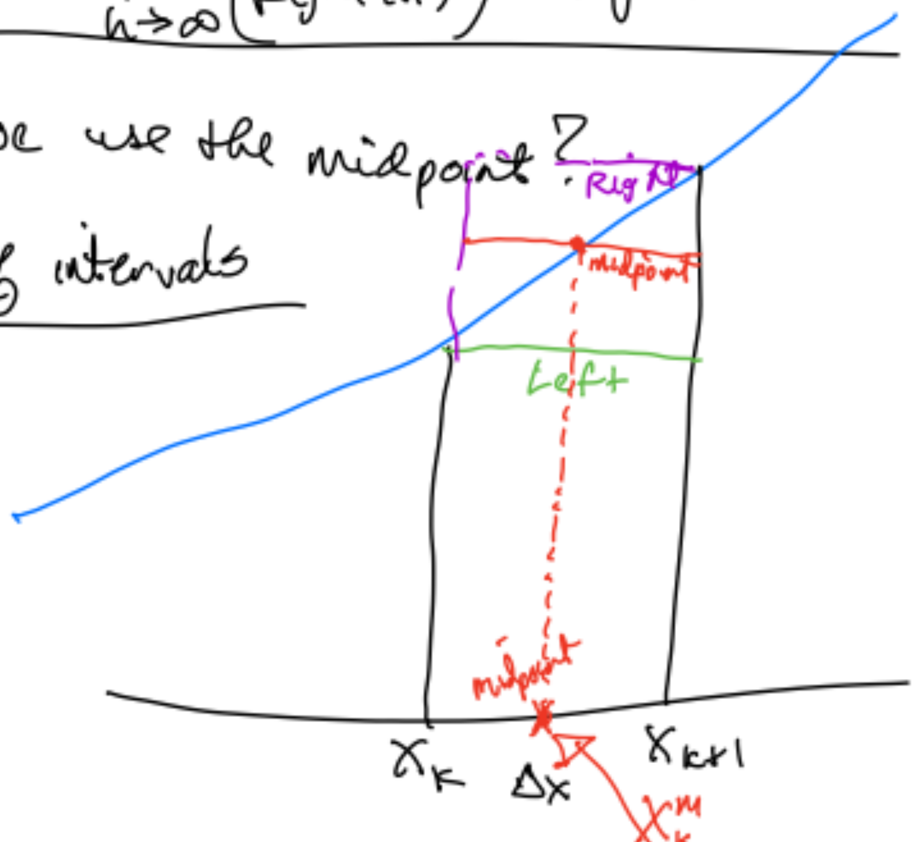
$$\text{Right}(n) = \sum_{k=1}^{n} f(x_k)\,\Delta x \text{, where } x_k = a + k\,\Delta x \ \forall k.$$

$$\Delta x = \frac{b-a}{n}, \ \cancel{x_j = a + j\,\Delta x}. \ \cancel{\forall j}$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\text{Right}(n)\right). \text{ again.}$$

What if we use the midpoint?

one of intervals



$$\frac{x_k + x_{k+1}}{2} = x_k + \frac{1}{2}\Delta x$$
$$= a + k\,\Delta x + \frac{1}{2}\Delta x$$

midpt $= a+\left(\frac{2k+1}{2}\right)\Delta x$

$$\text{Mid}(n) = \sum_{k=0}^{n-1} f\left(a+\left(\frac{2k+1}{2}\right)\Delta x\right)\cdot\Delta x,$$

where $\Delta x = \frac{b-a}{n}$

Midpoint approx to $\int_a^b f(x)\,dx$ with n-subdivisions

Example $\int_0^1 e^{-x^2}\,dx = ?$

$$\text{Left}(n) = \sum_{k=0}^{n-1} f(x_k)\Delta x \qquad x_k = a + k\Delta x$$

$\Delta x = \frac{b-a}{n}$

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$x_k = a + k\Delta x = 0 + k\frac{1}{n} = \frac{k}{n}$$

$$f(x_k) = e^{-x_k^2} = e^{-\left(\frac{k^2}{n^2}\right)}$$

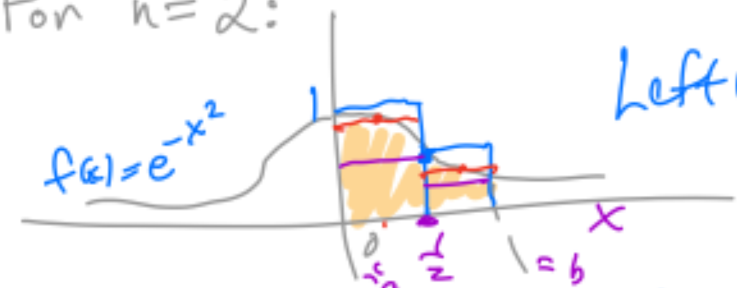
$$\text{Left}(n) = \sum_{k=0}^{n-1} e^{-k^2/n^2}\frac{1}{n}$$

$$\text{Right}(n) = \sum_{k=1}^{n} e^{-k^2/n^2}\frac{1}{n}.$$

Midpoint $x_k^M = a+\left(\frac{2k+1}{2}\right)\Delta x = \frac{2k+1}{2}\left(\frac{1}{n}\right) = \frac{2k+1}{2n}$

$$\text{Mid}(n) = \sum_{k=0}^{n-1} e^{-\left(\frac{2k+1}{2n}\right)^2} \frac{1}{n}$$

For $n=2$:



$$\text{Left}(2) = 1\cdot\frac{1}{2} + e^{-(1/2)^2}\cdot\frac{1}{2} = \frac{1+e^{-1/4}}{2}$$

$$\text{Right}(2) = e^{-(1/2)^2}\cdot\frac{1}{2} + e^{-1}\cdot\frac{1}{2} = \frac{e^{-1/4}+e^{-1}}{2}$$

$$\text{Mid}(2) = e^{-(1/4)^2}\cdot\frac{1}{2} + e^{-(3/4)^2}\cdot\frac{1}{2} = \frac{e^{-1/16}+e^{-9/16}}{2}$$

$$\text{Mid}(2) = \sum_{k=0}^{1} e^{-\left(\frac{2k+1}{4}\right)^2} \frac{1}{2} = \underbrace{e^{-\left(\frac{1}{4}\right)^2} \frac{1}{2}}_{k=0} + \underbrace{e^{-\left(\frac{3}{4}\right)^2} \frac{1}{2}}_{k=1}.$$

How close are these #s?

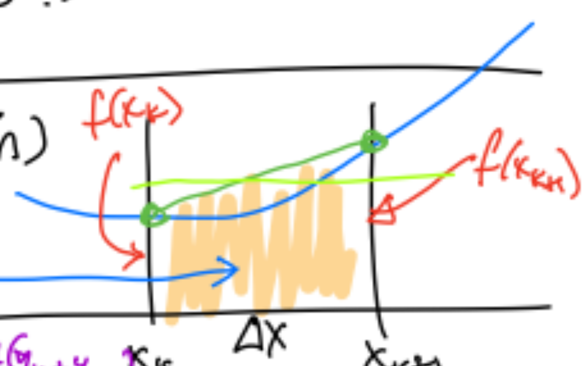
$\text{Left}(2) \approx .8894$

$\text{Right}(2) \approx .5733$

$\text{Mid}(2) \approx .7546$

$\int_0^1 e^{-x^2}dx = 0.7468\ldots$

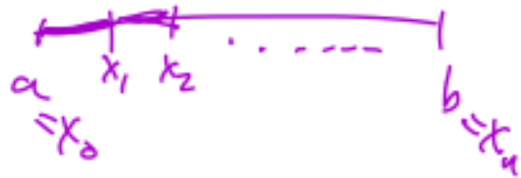
TRAPEZOID Rule $\text{TRAP}(n)$



Let $y_k = f(x_k)\ \forall k$.

Area $= \frac{1}{2}(f(x_k)+f(x_{k+1}))\Delta x = \frac{1}{2}(y_k+y_{k+1})\Delta x$

$\mathrm{Trap}(n) =$



$$\tfrac{1}{2}(y_0+y_1)\Delta x + \tfrac{1}{2}(y_1+y_2)\Delta x + \tfrac{1}{2}(y_2+y_3)\Delta x + \cdots + \tfrac{1}{2}(y_{n-1}+y_n)\Delta x$$

$$= \tfrac{1}{2} y_0 \Delta x + \tfrac{1}{2} y_1 \Delta x + \tfrac{1}{2} y_1 \Delta x + \tfrac{1}{2} y_2 \Delta x + \cdots + \tfrac{1}{2} y_{n-1}\Delta x + \tfrac{1}{2} y_n \Delta x$$

$$\mathrm{Trap}(n) = \left(\tfrac{1}{2} y_0 + y_1 + y_2 + \cdots + y_{n-1} + \tfrac{1}{2} y_n\right)\Delta x$$

$$\mathrm{Trap}(n) = \tfrac{1}{2}\left(\mathrm{Left}(n) + \mathrm{Right}(n)\right) \;!$$

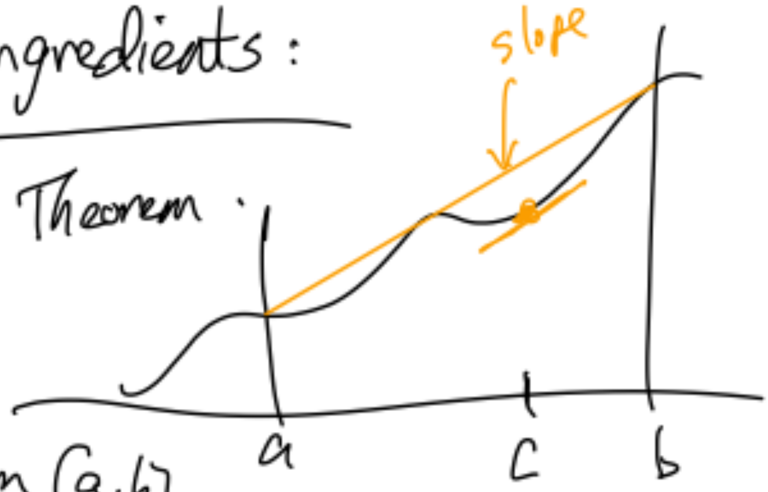
(To see this: $\tfrac{1}{2}\big(y_0\Delta x + y_1\Delta x + \cdots + y_{n-1}\Delta x$
$\qquad y_1\Delta x + \cdots + y_{n-1}\Delta x + y_n \Delta x\big)$

$$= \tfrac{1}{2}\left(y_0 + 2y_1 + \cdots + 2y_{n-1} + y_n\right)\Delta x.$$

How can we figure out how much error we have in these calculations?

Important Ingredients:

① Mean Value Theorem.



$F(x)$ continuous on $[a,b]$,
F is differentiable on (a,b),
then $\exists\, c \in (a,b)$ s.t.

$$\frac{F(b)-F(a)}{b-a} = F'(c).$$

$$F(b) = F(a) + F'(c)(b-a)$$

$$F(x) = F(a) + F'(c)(x-a)$$

Same as Lagrange Remainder formula for Taylor polynomial of degree 0.

Generalization: Newton Finite Difference formula with Remainder. F is $(n+1)$-diff'ble on $[a,b]$, $a = x_0$, $b = x_n$, $\Delta x = \frac{b-a}{n}$,
$x_k = a + k\Delta x$.

Let $y_j = F(x_j)$, $\Delta y_j = y_{j+1} - y_j$, etc $\Delta^2 y_j$

$$F(x) = y_0 + \frac{\Delta y_0}{h}(x-x_0) + \frac{1}{2!}\frac{\Delta^2 y_0}{h^2}(x-x_0)(x-x_1)$$

$$\cdots \frac{1}{n!}\frac{\Delta^n y_0}{h^n}(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

$$+ \frac{1}{\cancel{n!}\,(n+1)!} F^{(n+1)}(c)\,(x-x_0)(x-x_1)\cdots(x-x_n)$$

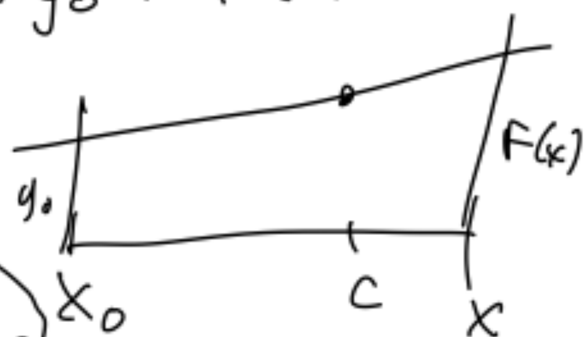
$\exists c \in (a,b)$ s.t. the equation above is true.

Note: the case $n=0$ of the above is

$$F(x) = y_0 + \frac{1}{1!}F'(c)(x-x_0)$$

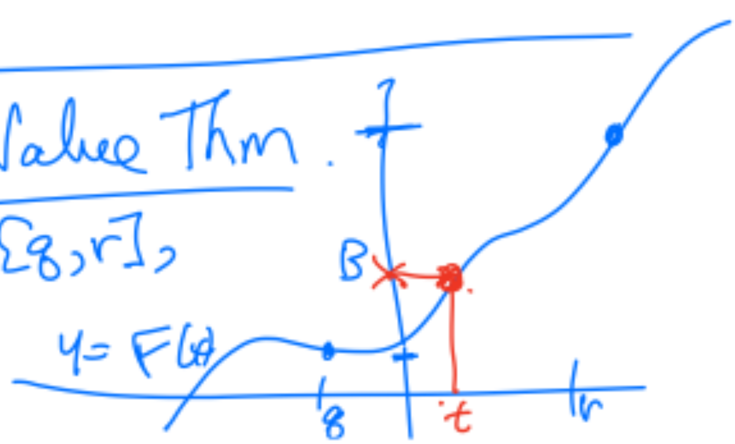
$$\Rightarrow F(x) = y_0 + F'(c)(x-x_0)$$

Same as Mean value Thm



Intermediate Value Thm.

If $F(x)$ is continuous $[q,r]$, and B is between



$F(q)$ & $F(r)$, then $\exists\, t \in [q, r]$
s.t. $F(t) = B$.